Estimating Costs of Children from Micro-Unit Records: A New Procedure Applied to Australian Data

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Abstract

Measuring the costs of children is of immense practical importance in a range of economic and social policy areas. In this paper, we introduce a new econometric procedure that improves on existing methods for obtaining estimates of such costs from a demand system. We develop, using an extended linear expenditure system, an iterative maximum likelihood estimator that overcomes possible estimation problems that arise from the 2-step estimation procedures employed by earlier authors. We also allow for a more general assumption about the equation "errors", that of non-zero correlation between the errors for different commodities in the same household. Another important contribution is the development of an estimation procedure for sets of seemingly unrelated regressions where the different sets of equations are linked by some common parameters. The proposed procedure is applied to the 1988-89 and 1993-94 Australian Household Expenditure Surveys and results obtained update estimates of both the commodity-specific and general scales previously obtained for Australia.

1. Introduction

Measuring the costs of children is of immense practical importance in a range of economic and social policy areas. When assessing the distribution of income, the progressiveness and effectiveness of tax and social security systems and the impact of government policies on living standards of households, it is necessary to examine the nature and level of these costs. In the economic literature, a conventional approach to estimating child costs is through the use of micro unit record data within the context of a utility framework. This approach yields child cost estimates (otherwise known as equivalence scales) that allow one to make direct comparisons between households of different sizes and composition. For example, a comparison of equivalence scales for households with and without children is a popular means of obtaining some representation of the costs that raising children imposes on a household. Indeed, it is the use of equivalence scales in income maintenance programs that results in larger benefits accruing to families with more and older children compared to families with fewer and younger children.

The calculation of household equivalence scales has a long and controversial history beginning with the pioneering work of Engel (1895) on Belgian working class expenditure data. The focus of the debate in more recent times centers on the legitimacy of making welfare comparisons based on "conditional" equivalence scales, which are scales derived from demand data and are computed "conditioned" on a predetermined demographic composition. It is argued that household welfare should be thought of as depending on a household composition directly as well as through the effects of household composition on commodity demands. "The expenditure level required to make a three-child family as well off as it would be with two children and $12,000 depends on how the family feels about children", wrote Pollak and Wales (1979). This argument led some authors to conclude that such scales are not useful for welfare comparisons. (See Browning (1992) and Nelson (1993) for a detailed overview of the identification problems of equivalence scales). Other authors, however, regard this as an overly negative assessment and counter claim that estimation of equivalence scales based on conditional preferences has a purposeful role in welfare comparisons. (e.g. Deaton and Muellbauer (1986), Blundell and Lewbel (1991), Nelson (1993)). Also, the relationship between the identifiability of scales and the testing of aggregation restrictions has been investigated by Lewbel (1989) and Nicol (1994).

It is not the purpose of this study to contribute to this on-going debate. Given that unconditional scales are not estimable at the present time, and that equivalence scales are in great demand for policy and welfare analysis, this study is developed based on the premise that equivalence scales from demand data are the best practicable approach to estimating costs of children. In this context, we introduce a new econometric procedure that improves on existing methods for estimating commodity-specific and general scales from an extended linear expenditure system. Specifically, we develop an iterative maximum likelihood estimator that overcomes possible estimation problems that arise from the 2-step estimation procedures employed by earlier authors. We also allow for a more general assumption about the equation "errors", that of non-zero correlation between the errors for different commodities in the same household. This assumption is more in line with that usually made for Engel functions and other systems of demand equations. Another important contribution is the development of an estimation procedure for sets of seemingly unrelated regressions where the different sets of equations are linked by some common parameters.

The proposed procedure is applied to the 1988-89 and 1993-94 Australian Household Expenditure Surveys and results obtained update estimates of both the commodity-specific and general scales previously obtained for Australia.

The paper is structured as follows. Section 2 describes the model and sets the notation. Section 3 summarises the estimation procedure and asymptotic covariance matrices; details of the derivation are left to an appendix of the paper. Section 5 applies the procedure to Australian data and a final section concludes.

2. The Model

The demand system employed here is the extended linear expenditure system (ELES) of Lluch (1973). The linearity assumption of this system is often questioned, as is the assumption of direct-additive utility from which the system is derived. However, the ELES was chosen for this study for a number of important reasons. First, the ELES is a convenient vehicle for carrying out relatively sophisticated research on consumer behavior even when available data on private consumption are limited. Since purely cross-section data generally give no price variation, inference about price effects requires strong theoretical specifications. Second, the ELES is chosen for its historical significance in equivalence scale research. ELES-based equivalence scales have been repeatedly estimated in the past and are particularly popular among researchers using Australian data. See, for example, Powell (1974, and references therein), Kakwani (1980), Binh and Whiteford (1990), Bradbury (1994), Valenzuela (1996) and Lancaster and Ray (1998). Using the ELES in this study facilitates comparison of results to these earlier ones. Thirdly, the ELES is used because of its simplicity. In this work, a new method of estimation of equivalence scales is derived. The ELES is ideal for this purpose because the system remains mathematically tractable but is still sufficiently complicated to warrant a number of econometric innovations. Once these innovations have been developed, they can be more readily applied to more complicated models at a later time. Special care was taken to split the sample into groups with few parametric restrictions on the scales and estimation was restricted to just eleven broad commodity groups, thereby mitigating the assumption of additive utility. The exercise is a natural starting point for demonstrating a new iterative procedure using maximum likelihood techniques. The study as a whole provides a useful addition to the available empirical evidence on equivalence scales in Australia. For examples of other more complex expenditure systems that have appeared in the literature see Pollak and Wales (1992) and Nicol (1994).

2.1 The Extended Linear Expenditure System

To describe the model, consider n commodity groups indexed by $i=1,2,\ldots,n$ and H types of households indexed by $h=1,2,..,H$ where household types are defined according to the number of adults and the number of children in the household. Define q_{ih} as the quantity of the i^{th} commodity consumed by the h-type household and s_{ih} is the i^{th} commodity-specific scale for the h-type household. The s_{ih} are factors used to adjust quantities q_{ih} in utility functions to show the effect of a change in the household's demographic composition on household utility and on specific commodity expenditures. On a per unit basis, a given q_{ih} provides less utility if it is shared with more people. How q_{ih} should be deflated to give the same per unit utility will depend on the commodity i and on the household type h. Thus the scale is subscripted with i and h. Utility is specified relative to a reference unit which is a household with two adults and no children; in this case we set $s_{ih}=1$. The q_{ih} in the utility functions for other household types are scaled by s_{ih} to give a comparable two-adult-zero-children utility function for all households.

Given this background, consider now the Klein-Rubin utility function where the consumption quantities q_{ih} are scaled as follows:

$$u_h = \sum_{i=1}^{n} b_i \ \ln\left(\frac{q_{ih}}{s_{ih}} - c_i \right) \tag{1}$$

where

b_i = is the marginal contribution to utility of the i^{th} commodity and satisfies the constraints $0 < b_i < 1$ and $\sum_{i=1}^{n} b_i = 1$;

c_i = is a parameter which, if interpreted as the subsistence quantity of the i^{th} commodity, satisfies the constraint $c_i > 0$.[1]

[1] Pollak and Wales (1992) prefer not to give c_i a strict subsistence interpretation letting negative values be a possibility.

Let p_i be the price of the i^{th} commodity and v_h be the total expenditure for the h-type household. Maximising the utility function (1) subject to the budget constraint $\sum_{i=1}^{n} p_i q_{ih} = v_h$ leads to the linear expenditure system (LES)

$$p_i q_{ih} = p_i s_{ih} c_i + b_i \left(v_h - \sum_{j=1}^{n} p_j s_{jh} c_j \right) \tag{2}$$

A household whose demand system is LES is often described as first purchasing "necessary", "subsistence" or "committed" quantities for each good ($s_{1h}c_1,\ldots,s_{nh}c_n$) and then dividing its remaining or "supernumerary" expenditure ($v_h - \sum_{i=1}^{n} p_i s_{ih} c_i$), among the goods in fixed proportions ($b_1,\ldots,b_n$). The system in (2) can be more compactly expressed as

$$v_{ih} = a_{ih} + b_i (v_h - a_h) \tag{3}$$

where

$v_{ih} = p_i q_{ih}$ is expenditure on the i^{th} commodity by the h-type household;

$a_{ih} = p_i s_{ih} c_i$ is subsistence expenditure for the i^{th} commodity and h-type household; and,

$a_h = \sum_{i=1}^{n} a_{ih}$ is the total subsistence expenditure for the h-type household.

The objective is to estimate a_{ih} and b_i with these estimates later being used to estimate the scales s_{ih}. Specifically, if $s_{ir} = 1$ denotes the scale for the reference household type, then

$$s_{ih} = \frac{p_i s_{ih} c_i}{p_i s_{ir} c_i} = \frac{a_{ih}}{a_{ir}} \tag{4}$$

However, without further information, not all of the a_{ih} are identified. The identification problem arises because, for a given household type, one of the n equations in (3) is redundant, redundancy being illustrated by summing both sides of (3) to yield

$$\sum_{ih}^{n} v_{ih} = \sum_{i=1}^{n} a_{ih} + \sum_{i=1}^{n} b_i (v_h - a_h) \tag{5}$$

or

$$v_h = a_h + (v_h - a_h) \tag{6}$$

The redundancy of one equation means that separate information is only available from n-1 equations. The problem is to estimate n intercept terms with only n-1 equations.

One solution to this identification problem is to include in the linear expenditure system in (3) a micro-consumption function given by

$$v_h = a_h + b(x_h - a_h) \tag{7}$$

where v_h is the total expenditure, x_h is net income, b is a common marginal propensity to consume for all households. This function shows that total expenditure v_h is composed of "committed" or "subsistence" expenditure a_h (the sum of the subsistence expenditures for each commodity) and a proportion b of "uncommitted" expenditure (x_h-a_h). The extended linear expenditure system or ELES is thus comprised of equations (3) and (7).

To estimate the parameters in ELES, Kakwani (1980) appended errors to these equations, and assumed the error variances can be different for each household type and for each commodity. He suggested first estimating a_h and b from (7) and then replacing a_h with its estimated equation in each of the commodity equations in (3). Then, to estimate a_h and b_i in (3), weighted least squares which allows for heteroskedasticity across different household types was applied to each of these equations. Using an external estimate of a_h identifies the remaining parameters.

The estimation procedure that is developed in this paper attempts to improve on Kakwani's procedure in two ways. First, because Kakwani estimated each of the commodity equations separately, he ignored any correlation that might exist between the errors that correspond to different commodity equations for a given household. Second, the '2-step' nature of the procedure ignored the effect of using estimates from one equation on the properties of the estimates from a second equation. An estimator which allows for error correlation across different commodity equations and which estimates all parameters simultaneously would seem more desirable. Specifically, as is demonstrated below, a common vector of coefficients for each household's system of equations means that one can improve upon equation-by-equation estimation despite the fact that a given household system has the same regressors in each equation.

To investigate how all the parameters might be jointly estimated, (7) is substituted into (3) to obtain

$$\begin{aligned} v_{ih} &= a_{ih} + b_i\left[\left(a_h + b\left(x_h - a_h\right)\right) - a_h\right] \\ &= \theta_{ih} + \eta_i x_h \end{aligned} \qquad (8)$$

where $\theta_{ih} = a_{ih} - b_i b a_h$ and $\eta_i = b_i b$.

Consider now the estimation of θ_{ih} and the η_i and how estimates of the structural parameters a_{ih}, b_i, b and a_h can be retrieved from these estimates. Given θ_{ih} and η_i, estimates of the structural parameters can be obtained using the expressions

$$b = \sum_{i=1}^{n} \eta_i \qquad (9)$$

$$a_h = \frac{\sum_{i=1}^{n} \theta_{ih}}{1 - \sum_{i=1}^{n} \eta_i} \qquad (10)$$

$$b_i = \frac{\eta_i}{\sum_{j=1}^{n} \eta_j} \qquad (11)$$

$$a_{ih} = \theta_{ih} + \frac{\eta_i \sum_{j=1}^{n} \theta_{jh}}{1 - \sum_{j=1}^{n} \eta_j} \qquad (12)$$

The system in (8) does not suffer from an identification problem because there are no redundant equations. All the n commodity equations for a given household type can be utilised.

2.2 Expressions for General Scales

A general equivalence scale s_h for the h-type household is defined as the ratio of income for that type of household to income of the reference household such that the indirect utility functions of the two household types are the same.

To obtain an expression for the general scales, we first consider the demand equations in (2). Dividing through by $p_i s_{ih}$ and using the result in equation (7), we get

$$\frac{q_{ih}}{s_{ih}} = c_i + \frac{b_i b}{p_i s_{ih}}\left(x_h - \sum_{j=1}^{n} p_j s_{jh} c_j \right) \tag{13}$$

Equation (13) is then substituted into the direct utility function in (1) and noting that $\sum_{i=1}^{n} b_i = 1$, we get

$$\begin{aligned} u_h &= \sum_{i=1}^{n} b_i \ \ln\left[\frac{b_i b}{p_i s_{ih}} \left(x_h - \sum_{j=1}^{n} p_j s_{jh} c_j \right) \right] \\ &= \ln b + \ln\left(x_h - \sum_{i=1}^{n} p_i s_{ih} c_i \right) + \sum_{i=1}^{n} b_i \ln b_i - \sum_{i=1}^{n} b_i \ln p_i - \sum_{i=1}^{n} b_i \ln s_{ih} \end{aligned} \tag{14}$$

For the standard reference household where s_{ir} = 1, the indirect utility function is thus expressed as

$$u_r = \ln b + \ln(x_r - \sum_{i=1}^{n} p_i c_i) + \sum_{i=1}^{n} b_i \ln b_i - \sum_{i=1}^{n} b_i \ln p_i \tag{15}$$

The general scale for the h-type household is given by the ratio of incomes $s_h = x_h / x_r$ that equates the two indirect utility functions. Working in this direction, we set $u_r = u_h$ and solve for x_h / x_r to obtain

$$\frac{x_h}{x_r} = \prod_{i=1}^{n} s_{ih}^{b_i} + \frac{1}{x_r}\left[\sum_{i=1}^{n} p_i s_{ih} c_i - \left(\prod_{i=1}^{n} s_{ih}^{b_i}\right)\left(\sum_{i=1}^{n} p_i c_i\right)\right] \tag{16}$$

Noting that $a_{ih}=p_i s_{ih} c_i$ and $a_h = \sum_{i=1}^{n} a_{ih}$ for all i, (16) can be equivalently written as

$$\begin{aligned} s_h = \frac{x_h}{x_r} &= \prod_{i=1}^{n} s_{ih}^{b_i} + \frac{1}{x_r}\left[\sum_{i=1}^{n} a_{ih} - \prod_{i=1}^{n} s_{ih}^{b_i} a_r\right] \\ &= \frac{a_h}{x_r} + \prod_{i=1}^{n} s_{ih}^{b_i}\left[1 - \frac{a_r}{x_r}\right] \end{aligned} \tag{17}$$

Along with the commodity specific scales s_{ih}, these general scales are the final quantities of interest. They capture the overall effect of a change in demographic composition on the total expenditure of the household. From (17), they are shown to be a function of the commodity specific scales s_{ih}'s and are calculated based on a chosen reference income level of the reference household. If we write the first term in the second line of (17) as $\frac{a_h}{a_r}\frac{a_r}{x_r}$ we can see that a general scale is a weighted average of the $\prod_{i=1}^{n} s_{ih}^{b_i}$ term and $\frac{a_h}{a_r}$, where the former is a weighted geometric mean of the s_{ih}'s and the latter is a ratio of relative subsistence costs.

3. Stochastic Assumptions and ML Estimation

Suppose now that there are M_h observations (households) with demographic composition type h. In the notation that follows, the symbols v_{ih} and x_h which previously represented scalar quantities for a given household, will become ($M_h \times 1$) vectors containing all observations on households of type h. Returning to equation (8), adding stochastic terms, the system we wish to estimate can be written as

$$\begin{bmatrix} v_{1h} \\ v_{2h} \\ \vdots \\ v_{nh} \end{bmatrix} = \begin{bmatrix} z_h & & & \\ & z_h & & \\ & & \ddots & \\ & & & z_h \end{bmatrix} \begin{bmatrix} \theta_{1h} \\ \theta_{2h} \\ \vdots \\ \theta_{nh} \end{bmatrix} + \begin{bmatrix} x_h & & & \\ & x_h & & \\ & & \ddots & \\ & & & x_h \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix} + \begin{bmatrix} e_{1h} \\ e_{2h} \\ \vdots \\ e_{nh} \end{bmatrix} \tag{18}$$

or

$$\mathbf{V}_h = \mathbf{Z}_h \boldsymbol{\Theta}_h + \mathbf{X}_h \boldsymbol{\eta} + \mathbf{E}_h \tag{19}$$

where

h = 1,2,…, H refers to household composition type h;

n = refers to the number of commodity groups;

v_{ih} is an $(M_h \times 1)$ vector of observations on expenditure for the i^{th} commodity and the h-type household;

z_h is an $(M_h \times 1)$ vector of ones;

x_h is an $(M_h \times 1)$ vector of observations on income for household type h;

e_{ih} is an $(M_h \times 1)$ vector of errors;

$\mathbf{V}_h$ is of dimension $(nM_h \times 1)$;

$\mathbf{Z}_h = I_n \otimes z_h$ is an $(nM_h \times n)$ matrix of dummy variables;

$\mathbf{X}_h = I_n \otimes x_h$ is an $(nM_h \times n)$ matrix matrix of household incomes;

$\boldsymbol{\Theta}_h, \boldsymbol{\eta}$ are $(n \times 1)$ vectors of unknown parameters; and

$\mathbf{E}_h$ is an $(nM_h \times 1)$ vector of errors which are assumed to be distributed as

$$\mathbf{E}_h \sim N\left[0,\ \boldsymbol{\Omega}_h \otimes I_{M_h}\right]. \tag{20}$$

Thus, the error covariance matrix $\boldsymbol{\Omega}_h$ is allowed to be different for different household types. Because $\boldsymbol{\Omega}_h$ is not diagonal, correlation between errors from equations for different commodities and the same household, is permitted. Zero error correlation is assumed across different households[2]. Thus, in addition to (20), $E(\mathbf{E}_h \mathbf{E}_k') = 0$ for $h \neq k$.

[2] The sample is assumed to be random.

The task is to derive expressions for the maximum likelihood estimators of $\boldsymbol{\Theta}_h$, $\boldsymbol{\Omega}_h$, and $\boldsymbol{\eta}$, as well as asymptotic covariance matrices for these estimates and asymptotic covariance matrices for the consequent maximum likelihood estimates for the parameters in equations (9) – (12). There are H seemingly unrelated regression systems, one for each household type, each comprising n equations, and with constraints on the parameters across different systems.

3.1 An Iterative Maximum Likelihood Estimator

The results derived in the appendix lead to the following convenient iterative procedure for computing maximum likelihood estimates:

1. Express v_{ih} and x_h in terms of deviations from their household-type means. That is, compute $v_{ih}^* = v_{ih} - \bar{v}_{ih} z_h$ and $x_h^* = x_h - \bar{x}_h z_h$ where $\bar{v}_{ih} = M_h^{-1} z_h' v_{ih}$ and $\bar{x}_h = M_h^{-1} z_h' x_h$.
2. Find the least squares estimates $\hat{\eta}_i^h = \left(x_h^{*\prime} x_h^* \right)^{-1} x_h^{*\prime} v_{ih}^*$ for i=1,2…, n and h = 1,2,…,H.
3. Find H initial estimates of the $\boldsymbol{\Omega}_h$ as[3]

$$\left[\hat{\boldsymbol{\Omega}}_h \right]_{ij} = \left(v_{ih}^* - x_h^* \hat{\eta}_i^h \right)' \left(v_{jh}^* - x_h^* \hat{\eta}_j^h \right) / M_h \tag{21}$$

4. Compute a pooled estimate for $\boldsymbol{\eta}$ as

$$\hat{\boldsymbol{\eta}} = \left[\sum_{h=1}^{H} \left(x_h^{*\prime} x_h^* \right) \hat{\boldsymbol{\Omega}}_h^{-1} \right]^{-1} \sum_{h=1}^{H} \left(x_h^{*\prime} x_h^* \right) \hat{\boldsymbol{\Omega}}_h^{-1} \hat{\boldsymbol{\eta}}^h \tag{22}$$

   where $\hat{\boldsymbol{\eta}}^h = \left(\hat{\eta}_1^h, \hat{\eta}_2^h, \ldots, \hat{\eta}_n^h \right)'$.

5. Repeat step (3) with $\hat{\eta}_i^h$ replaced by $\hat{\eta}_i$ that is computed from (22).
6. Repeat steps (4) and (5) until convergence.
7. Compute estimates of the θ_{ih} from $\hat{\theta}_{ih} = \bar{v}_{ih} - \bar{x}_{ih} \hat{\eta}_i$.

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[3] Note that steps (2) and (3) can be computed at the same time with H seemingly unrelated regressions of each of $\left(v_{1h}^*, v_{2h}^*, \ldots, v_{nh}^* \right)$ on x_h^*, with no constant.

Maximum likelihood estimates of the original parameters can then be computed from equations (9) – (12).

3.2 Asymptotic Covariance Matrices

For inference purposes, the asymptotic covariance matrices for $\hat{\mathbf{\Theta}}_h$ and $\hat{\boldsymbol{\eta}}$, as well as those for the estimators of the original parameters *b, a_h, b_i* and a_{ih}, are of interest. Using V(.) to denote an asymptotic variance or covariance matrix, in the appendix we show that

$$V(\hat{\mathbf{\Theta}}_h) = M_h^{-1}\mathbf{\Omega}_h + \bar{x}_h^2 D^{-1} \tag{23}$$

and

$$V(\hat{\boldsymbol{\eta}}) = D^{-1} \tag{24}$$

where $D = \sum_{h=1}^{H}\left[\left(x_h' x_h - M_h \bar{x}_h^2\right)\mathbf{\Omega}_h^{-1}\right]$. Further, let $z = (1,1,\ldots,1)'$, $\mathbf{B} = (b_1, b_2, \ldots, b_n)'$, $\boldsymbol{\alpha}_h = \left(a_{1h}, a_{2h}, ..., a_{nh}\right)'$, $C = I_n - \frac{1}{b}\boldsymbol{\eta} z'$ and $C^* = I_n + \frac{1}{1-b}\boldsymbol{\eta} z'$. Then, also from the appendix, we have

$$V(\hat{\mathbf{B}}) = \frac{1}{b^2} C D^{-1} C' \tag{25}$$

$$V(\hat{b}) = z' D^{-1} z \tag{26}$$

$$V(\hat{\boldsymbol{\alpha}}_h) = C^*\left[M_h^{-1}\mathbf{\Omega}_h + \left(\bar{x}_h - a_h\right)^2 D^{-1}\right] C^{*\prime} \tag{27}$$

and

$$V(\hat{a}_h) = z' C^*\left[M_h^{-1}\mathbf{\Omega}_h + \left(\bar{x}_h - a_h\right)^2 D^{-1}\right] C^{*\prime} z\,. \tag{28}$$

From these results, we show that the variance of the estimator of a commodity specific scale is given by

$$V(\hat{s}_{ih}) = \frac{1}{a_{ir}^2} V\left(\hat{a}_{ih}\right) + \frac{a_{ih}^2}{a_{ir}^4} V\left(\hat{a}_{ir}\right) - \frac{2a_{ih}}{a_{ir}^3}\operatorname{cov}\left(\hat{a}_{ih}, \hat{a}_{ir}\right) \tag{29}$$

where $\operatorname{cov}(\hat{a}_{ih}, \hat{a}_{ir})$ is the i^{th} diagonal element of

$$\operatorname{cov}\left(\hat{\boldsymbol{\alpha}}_h, \hat{\boldsymbol{\alpha}}_r\right) = \left(\overline{x}_h - a_h\right)\left(\overline{x}_r - a_r\right) C^* D^{-1} C^{*\prime} \quad . \tag{30}$$

4. Empirical Application

The iterative procedure described in Section 3.1 of this paper is applied to household unit record data from the 1993-94 Household Expenditure Survey of the Australian Bureau of Statistics. The public-use tapes contain data from a total of 8390 households representing over 5.4 million Australian households from all over the country.

In our analysis, we grouped commodities into eleven broad private expenditure categories. They are: Housing, Fuel and Power, Food, Alcohol and Tobacco, Clothing and Footwear, Household Furnishings and Equipment, Medical and Health Care, Transport, Recreation and Entertainment, Personal Care, and Others[4]. Further, households were restricted to those of related persons with one or two adults and at most three children, resulting in eight household types. Adults are all persons aged 17 or older and children refer to all those aged 16 or younger. Households not belonging to any of these types are excluded. Information from some 300 households were also discarded because of reported negative expenditures on certain items. These observations[5] were not consistent with the economic model set up in Section 2.

All exclusions leave 6752 households for econometric analysis. Of these, 37 percent were of the type (2,0) where the first number in the bracket refers to the number of adults, and the second number refers to the number of children (see Table 1). Further, 25 percent were of the type (1,0). This implies that 62 percent of the total households in the sample are without children. These households are mostly in the older age groups (household head typically 45 years old or older) and are inferred to have children who are already living away and/or financially independent. Meanwhile, the households with children

[4] These classification was chosen to facilitate comparison with previous studies using Australian data. The detailed composition of these commodity groups can be obtained from the authors on request.
[5] Negative expenditures were observed mostly on Transport, and also on Recreation and Entertainment.

tended to belong to the younger age bands (household head aged between 25 and 40). On average, two-adult households have higher weekly incomes compared to one-adult households, and households with children have higher incomes than those without. The differences in these income levels are more pronounced across households with two-adults than across those with one adult.

Application of the iterative method yielded estimates of θ_{ih} and η_i. These estimates[6] do not carry a direct economic interpretation but are important to the procedure as they lead to the estimation of the marginal propensities and subsistence expenditures. We also note that Steps 4 and 5 of the procedure did not take long to converge (5th iteration); the procedure, by and large, yielded estimates with very small standard errors.

Table 2 presents the marginal budget shares b_i for each expenditure category; it also presents estimates of subsistence expenditures a_{ih} for each expenditure category and household type. In general, the subsistence expenditures increase with household size, with wider differentials occurring across two-adult households compared to one-adult households. With the exception of one-member households, expenditure on Food was on top of the shopping list, followed closely by Housing, then Transport, and then Household furnishings. Together, these items make up between 60 to 73 per cent of subsistence expenditures of a typical Australian household.

Table 3 presents the estimates of commodity-specific scales. A two-adult household with no children is chosen to be the reference household for which s_{ih} is set to 1. The scale value of 1.38 for housing of a (2,1) household means that this type of household needs a housing expenditure 38 percent more than the typical (2,0) household to maintain the same standard of living as the latter. Similarly, the Fuel and power scale of 0.68 for the (1,0) type household implies that a single-adult household needs to consume more than half the Fuel and power requirements of a typical childless couple if it is to be on a comparable standard of living.

For most commodities, the commodity scales increase with the increase in the number of children. These increases are observed to occur at a decreasing rate, indicating economies of scale for additional children. After the first child, there exists strong economies of scale for additional children, particularly for expenditures towards Housing, Fuel and power, and Household furnishings and Transport. For two-adult households, the decline in the Housing scale after the first child is unexpected; given the magnitude of the standard errors, this decline could be attributable to sampling error.

The magnitudes of the scales for Alcohol and tobacco decline significantly as the number of children in the household increases. Also, the scales for Medical and health care, Recreation and entertainment, and Others commodity groups exhibit no defined trend for one-adult households. A more thorough investigation of expenditure patterns of households may be required for us to provide definitive explanations for such deviations but one possibility is that the presence of children in the household tends to influence expenses away from 'adult goods' under which alcohol, tobacco and other miscellaneous goods are classified.

Has there been a significant change in the scale relativities over time? Information from Tables 4 and 5, which compare some estimates for the survey years 1984, 1988-89 and 1993-94 provide some answers. In Table 5, commodity-specific scale estimates calculated from the parameter estimates of Binh and Whiteford (1990) that used the 1984 data are presented and compared with our estimated scales based on 1988-89 and 1993-94 data. Since it could be argued that a difference in results may be attributable to the new maximum likelihood procedure, rather than to a change in consumption patterns, estimates from the 1988 data set obtained using Kakwani's estimation method (the procedure used by Binh and Whiteford (1990)) are also presented. The two sets of 1988-89 scales are very similar with no one method exhibiting consistently higher or lower

[6] Not shown here for space reasons but can be made available if requested.

values. The estimated standard errors for both sets[7] show more divergence, but again do not display any consistent over or underestimation. There are noticeable changes in the scales over time. For the one-adult households with children, the estimated scales for Housing, Fuel and power, Food, Medical and health care, Transport and Others have typically declined between 1984 and 1988-89 and then increased again between 1988-89 and 1993-94. In contrast, there have been some declines in the estimated scales for the two-adult households for these same commodities.

Overall, the direction of the change is the same for the (2,1) and (2,2) household types while the change in the scale values for the (2,3) household type tends to be in the other direction. Interestingly, the only consistent (direction of) change for all household types was observed for Alcohol and tobacco. For this commodity, scales decreased significantly from 1984 to 1988-89 and the corresponding estimates for the 1993-94 scales registered larger increases for larger size households. From these results, it would seem that the presence of children has a deterring effect on the consumption of Alcohol and tobacco.

It is also noted that there are fewer economies of scale in Housing in the 1988-89 data set, but greater economies of scale in Food. The largest differences in the scale estimates occurred in the one-adult, three-children household groups. Since the number of households in this group is relatively small, and the standard errors of the estimated scales are relatively high, these differences may reflect sampling error.

The general scales computed from equation (17) are presented in Table 5. Because these scales depend on income x_r, they are computed for three income levels[8]. Also presented in Table 5 are three estimates of each scale – each one corresponding to the 1984, 1988-89 and 1993-94 survey years. First noted is that the estimated general scales are stable over different reference income levels. Also, for two-adult households, the 1993-94

[7] Not shown here for space reasons but can be provided by the authors on request.
[8] Levels were made comparable to those used by Binh and Whiteford (1990) to facilitate comparison.

scales are less than both the 1984 and 1988-89 scales. The 1988-89 and 1993-94 estimates are generally quite similar. Further, the conclusion by Binh and Whiteford (1990) that "there is strong evidence of economies of scale in the second child but adding the third child increased these households' needs considerably" no longer holds for the later data sets. The 1993-94 scales, in particular, suggest equal cost requirements for the 2^{nd} and 3^{rd} children.

6 Conclusion

We have introduced a maximum likelihood estimation procedure for an extended linear expenditure system that has different intercepts and different error covariance matrices for groups of households with different compositions. Estimates of household equivalence scales, both general and commodity-specific, can be obtained from this system. The procedure was applied to data from the 1993-94 Australian HES. For this particular case, the maximum likelihood estimates were similar to those obtained from a two-step estimation procedure originally due to Kakwani (1980). However, with respect to the methodology, the maximum likelihood procedure is an improvement over that of Kakwani, and we cannot conclude that the two techniques will yield similar estimates in other samples. Furthermore, the general error covariance assumptions make it likely that the standard errors will more accurately assess the reliability of estimation. A comparison of estimates with those obtained using data from three survey years uncovered some changes in the equivalence scales which are likely to be useful for government welfare-payment programs.

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Table 1. Sample Characteristics, HES 1993-94.

	Household Type (no. of adults, no. of children)							
	(1,0)	*(1,1)*	*(1,2)*	*(1,3)*	*(2,0)*	*(2,1)*	*(2,2)*	*(2,3)*
Sample Size	1702	192	149	60	2509	690	845	425
Age of Household Head	53.03	34.08	33.69	30.58	48.12	33.1	35.46	35.04
Average Weekly Household Income	281.89	339.99	377.83	346.83	547.18	612.54	634.78	632.85
	(227.89)	(162.32)	(151.71)	(100.60)	(345.90)	(333.63)	(349.07)	(365.70)
Average Weekly Household Expenditure	319.23	404.52	468.86	440.3	593.52	695.37	739.92	762.18
	(232.00)	(218.53)	(278.42)	(253.24)	(347.72)	(345.04)	(370.39)	(390.11)

Note: Standard errors are in parentheses.

Table 2. Parameters Estimates of Marginal Propensities and Subsistence Expenditures. Australia 1993-94.

Commodity Type		*Subsistence Expenditures (a_{ih}'s)* *Household Type (no. of adults, no. of children)*							
	b_i	*(1,0)*	*(1,1)*	*(1,2)*	*(1,3)*	*(2,0)*	*(2,1)*	*(2,2)*	*(2,3)*
Housing	0.1742	58.9867	92.4664	86.0663	93.7371	72.0887	99.4455	94.3581	91.7331
	(0.0033)	(1.5850)	(5.0680)	(5.5520)	(7.9753)	(1.7958)	(4.0119)	(3.4111)	(4.4120)
Fuel & Power	0.0089	10.5317	14.5945	17.3831	19.8280	15.5426	19.1783	20.8313	21.2057
	(0.0004)	(0.1935)	(0.6168)	(0.8775)	(1.6085)	(0.2185)	(0.4735)	(0.4056)	(0.5296)
Food	0.1144	49.2828	70.0878	90.9117	93.8108	94.3576	114.7908	136.1056	153.2327
	(0.0021)	(0.9224)	(2.7902)	(3.6088)	(4.9965)	(1.1158)	(2.3024)	(2.3963)	(3.5558)
Alcohol & Tobacco	0.0267	15.0098	15.0217	13.0949	15.6954	27.0314	26.2622	22.2245	22.1742
	(0.0012)	(0.6296)	(1.3700)	(1.3927)	(2.1010)	(0.7533)	(1.2903)	(1.0745)	(1.5231)
Clothing & Footwear	0.0798	11.5402	19.3199	27.9234	27.2097	24.4825	27.2869	38.0946	39.6102
	(0.0020)	(0.8750)	(2.2997)	(3.5633)	(5.1879)	(1.2075)	(1.8204)	(2.4502)	(4.2270)
Household Furnishings	0.1118	32.0272	52.0984	58.1885	56.8986	66.3583	81.3064	82.7068	76.0864
& Equipment	(0.0033)	(1.2990)	(3.7832)	(5.1105)	(9.7928)	(2.2263)	(4.1964)	(4.0728)	(4.4750)
Medical & Health Care	0.0404	12.5725	13.0920	16.7215	12.9669	25.2899	27.9016	32.1832	31.0671
	(0.0011)	(0.5075)	(1.3643)	(1.9228)	(2.7447)	(0.6022)	(1.4535)	(1.1503)	(1.5508)
Transport	0.1382	44.5204	54.7218	61.2014	68.1301	84.1265	101.4519	95.0724	102.5578
	(0.0051)	(2.5005)	(8.5809)	(6.7731)	(14.8432)	(2.8015)	(5.7070)	(5.2991)	(7.3800)
Recreation	0.1770	38.3862	43.7980	56.1104	50.0688	75.6770	71.7307	77.2948	92.9126
& Entertainment	(0.0041)	(1.8603)	(5.0096)	(6.5329)	(10.7139)	(2.6125)	(4.2536)	(3.8692)	(6.3789)
Personal Care	1.37E-02	5.2558	7.3174	7.5513	5.5752	10.3200	9.8865	11.8390	12.2997
	(0.0006)	(0.2605)	(0.7996)	(0.7734)	(0.9157)	(0.3362)	(0.6148)	(0.5695)	(0.8062)
Others	0.1149632	18.3025	25.4807	50.3898	29.9269	33.3312	43.2878	51.5687	63.9146
	(0.0029)	(1.2940)	(2.5045)	(14.6067)	(6.1602)	(1.9183)	(3.4159)	(2.8845)	(4.5032)
Total	1.0000	296.4158	407.9986	485.5424	473.8474	528.6056	622.5285	662.2790	706.7942

Note: The estimated standard errors are in parentheses.

Table 3. Estimates of Commodity-Specific Scales, Australia 1993-94.

Commodity Type	*Commodity Specific Scales Household Type (no. of adults, no. of children)*							
	(1,0)	(1,1)	(1,2)	(1,3)	(2,0)	(2,1)	(2,2)	(2,3)
Housing	0.82	1.28	1.19	1.30	1.00	1.38	1.31	1.27
	(0.04)	(0.10)	(0.10)	(0.14)	(0.04)	(0.08)	(0.07)	(0.09)
Fuel & Power	0.68	0.94	1.12	1.28	1.00	1.23	1.34	1.36
	(0.03)	(0.06)	(0.09)	(0.16)	(0.03)	(0.06)	(0.05)	(0.06)
Food	0.52	0.74	0.96	0.99	1.00	1.22	1.44	1.62
	(0.03)	(0.06)	(0.08)	(0.11)	(0.04)	(0.06)	(0.07)	(0.09)
Alcohol & Tobacco	0.56	0.56	0.48	0.58	1.00	0.97	0.82	0.82
	(0.06)	(0.10)	(0.10)	(0.15)	(0.09)	(0.11)	(0.09)	(0.12)
Clothing & Footwear	0.47	0.79	1.14	1.11	1.00	1.11	1.56	1.62
	(0.11)	(0.24)	(0.36)	(0.48)	(0.19)	(0.24)	(0.33)	(0.45)
Household Furnishings	0.48	0.79	0.88	0.86	1.00	1.23	1.25	1.15
& Equipment	(0.06)	(0.14)	(0.18)	(0.31)	(0.11)	(0.17)	(0.16)	(0.17)
Medical & Health Care	0.50	0.52	0.66	0.51	1.00	1.10	1.27	1.23
	(0.06)	(0.12)	(0.16)	(0.22)	(0.09)	(0.15)	(0.14)	(0.16)
Transport	0.53	0.65	0.73	0.81	1.00	1.21	1.13	1.22
	(0.08)	(0.20)	(0.17)	(0.34)	(0.12)	(0.18)	(0.17)	(0.21)
Recreation	0.51	0.58	0.74	0.66	1.00	0.95	1.02	1.23
& Entertainment	(0.07)	(0.14)	(0.18)	(0.29)	(0.12)	(0.14)	(0.14)	(0.20)
Personal Care	0.51	0.71	0.73	0.54	1.00	0.96	1.15	1.19
	(0.07)	(0.17)	(0.16)	(0.18)	(0.12)	(0.15)	(0.15)	(0.19)
Others	0.55	0.76	1.51	0.90	1.00	1.30	1.55	1.92
	(0.10)	(0.17)	(0.82)	(0.36)	(0.16)	(0.25)	(0.25)	(0.35)

Note: The estimated standard errors are in parentheses.

Table 4 Estimates of Commodity-Specific Scales

		Commodity-Specific Scales (S_{ih}) Household Type (no. of adults, no. of children)							
Commodity Type	***Year****	***(1,0)***	***(1,1)***	***(1,2)***	***(1,3)***	***(2,0)***	***(2,1)***	***(2,2)***	***(2,3)***
Housing	1984	0.80	1.22	1.23	1.24	1.00	1.51	1.52	1.51
	1988-89a	0.80	0.97	1.09	1.20	1.00	1.47	1.50	1.60
	1988-89b	0.82	1.03	1.15	1.28	1.00	1.49	1.52	1.65
	1993-94	0.82	1.28	1.19	1.30	1.00	1.38	1.31	1.27
Fuel & Power	1984	0.61	0.99	1.02	1.38	1.00	1.23	1.32	1.50
	1988-89a	0.67	0.90	1.04	1.08	1.00	1.19	1.34	1.42
	1988-89b	0.67	0.92	1.06	1.11	1.00	1.21	1.34	1.44
	1993-94	0.68	0.94	1.12	1.28	1.00	1.23	1.34	1.36
Food	1984	0.51	0.65	0.95	1.26	1.00	1.17	1.40	1.61
	1988-89a	0.53	0.71	0.93	1.04	1.00	1.23	1.42	1.56
	1988-89b	0.53	0.73	0.94	1.06	1.00	1.24	1.42	1.58
	1993-94	0.52	0.74	0.96	0.99	1.00	1.22	1.44	1.62
Alcohol & Tobacco	1984	0.58	0.64	0.74	0.80	1.00	1.15	1.11	1.15
	1988-89a	0.57	0.46	0.39	0.34	1.00	0.95	0.87	0.76
	1988-89b	0.57	0.46	0.39	0.34	1.00	0.95	0.86	0.76
	1993-94	0.56	0.56	0.48	0.58	1.00	0.97	0.82	0.82
Clothing & Footwear	1984	0.38	0.94	1.46	2.18	1.00	1.15	1.32	1.68
	1988-89a	0.53	0.89	0.90	1.38	1.00	1.27	1.39	1.63
	1988-89b	0.53	0.91	0.92	1.40	1.00	1.28	1.40	1.64
	1993-94	0.47	0.79	1.14	1.11	1.00	1.11	1.56	1.62
Household Furnishings	1984	0.46	0.85	1.05	1.24	1.00	1.14	1.20	1.37
& Equipment	1988-89a	0.56	0.71	0.83	0.89	1.00	1.48	1.14	1.37
	1988-89b	0.55	0.66	0.77	0.81	1.00	1.45	1.15	1.32
	1993-94	0.48	0.79	0.88	0.86	1.00	1.23	1.25	1.15
Medical & Health Care	1984	0.47	0.33	0.43	0.63	1.00	1.15	1.20	1.36
	1988-89a	0.53	0.44	0.64	0.47	1.00	1.25	1.28	1.28
	1988-89b	0.54	0.47	0.68	0.51	1.00	1.26	1.28	1.31
	1993-94	0.50	0.52	0.66	0.51	1.00	1.10	1.27	1.23
Transport	1984	0.46	0.62	0.75	1.19	1.00	1.34	1.16	1.32
	1988-89a	0.53	0.61	0.66	0.85	1.00	1.01	1.19	1.41
	1988-89b	0.52	0.57	0.62	0.78	1.00	1.02	1.19	1.37
	1993-94	0.53	0.65	0.73	0.81	1.00	1.21	1.13	1.22
Recreation	1984	0.53	0.62	0.68	1.01	1.00	0.92	1.15	1.19
& Entertainment	1988-89a	0.54	0.61	0.54	0.88	1.00	1.03	1.28	1.40
	1988-89b	0.54	0.58	0.51	0.82	1.00	1.03	1.28	1.37
	1993-94	0.51	0.58	0.74	0.66	1.00	0.95	1.02	1.23
Personal Care	1984	0.62	0.69	1.33	1.32	1.00	1.19	1.13	1.22
	1988-89a	0.54	0.76	0.95	0.71	1.00	1.19	1.29	1.17
	1988-89b	0.54	0.78	0.97	0.73	1.00	1.19	1.29	1.17
	1993-94	0.51	0.71	0.73	0.54	1.00	0.96	1.15	1.19
Others	1984	0.45	1.34	0.99	1.98	1.00	1.19	1.26	1.86
	1988-89a	0.57	1.04	0.90	0.82	1.00	1.40	1.80	2.10
	1988-89b	0.57	1.02	0.89	0.81	1.00	1.39	1.79	2.08
	1993-94	0.55	0.76	1.51	0.90	1.00	1.30	1.55	1.92

*1984 scales derived from the ELES parameters estimates presented in Binh and Whiteford (1990) which used 1984 HES data. 1988-89a scales derived using the Kakwani procedure applied to 1988-89 HES data (own calculations). 1988-89b and 1993-94 scales derived using the proposed MLE procedure applied to 1988-89 and 1993-94 HES data respectively (own calculations).

Table 5 Estimates of General Scales

*Reference Income***	*Year**	*General Scales (S_h)* *Household Type (no. of adults, no. of children)* (1,0)	(1,1)	(1,2)	(1,3)	(2,0)	(2,1)	(2,2)	(2,3)
Low Income ($325 p.w.)	**1984**	0.53	0.80	0.95	1.27	1.00	1.20	1.28	1.44
	1988-89a	0.59	0.73	0.82	0.94	1.00	1.23	1.33	1.46
	1988-89b	0.58	0.72	0.81	0.92	1.00	1.23	1.32	1.45
	1993-94	0.56	0.78	0.91	0.91	1.00	1.18	1.25	1.33
Medium Income ($450 p.w.)	**1984**	0.52	0.81	0.94	1.28	1.00	1.20	1.27	1.44
	1988-89a	0.58	0.73	0.80	0.92	1.00	1.23	1.32	1.47
	1988-89b	0.58	0.72	0.80	0.91	1.00	1.23	1.33	1.46
	1993-94	0.56	0.77	0.92	0.90	1.00	1.18	1.25	1.34
High Income ($700 p.w.)	**1984**	0.52	0.81	0.94	1.29	1.00	1.19	1.26	1.45
	1988-89a	0.58	0.73	0.78	0.92	1.00	1.23	1.32	1.47
	1988-89b	0.58	0.72	0.79	0.91	1.00	1.23	1.33	1.47
	1993-94	0.56	0.77	0.92	0.89	1.00	1.18	1.25	1.34

* 1984 Scales reprinted from Binh and Whiteford (1990) which used 1984 Household Expenditure Survey data.
1988-89a scales derived using the Kakwani procedure applied to 1988-89 HES data (own calculations).
1988-89b and 1993-94 scales derived using the proposed MLE procedure applied to 1988-89 and 1993-94 HES data respectively (own calculations).
**The scales have been evaluated using the listed incomes as reference levels.

Appendix

Derivation of Maximum Likelihood Estimators

Noting that $\mathbf{V}_1$, $\mathbf{V}_2$, …, $\mathbf{V}_H$ in the system of equations defined in (19) are independent, the log-likelihood for all parameters, given data on all household types, can be written as

$$\begin{aligned}\log \mathrm{L} = &\frac{nM}{2}\log(2\pi) - \frac{1}{2}\sum_{h=1}^{H} M_h \log\left|\boldsymbol{\Omega}_h\right| \\ &-\frac{1}{2}\sum_{h=1}^{H}\left(\mathbf{V}_h - \mathbf{Z}_h\boldsymbol{\Theta}_h - \mathbf{X}_h\boldsymbol{\eta}\right)'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\left(\mathbf{V}_h - \mathbf{Z}_h\boldsymbol{\Theta}_h - \mathbf{X}_h\boldsymbol{\eta}\right)\end{aligned} \tag{A1}$$

where $M = \sum_{h=1}^{H} M_h$. To maximise this function, we first concentrate out the $\boldsymbol{\Theta}_h$. Working in this direction, the last term can be written (without the summation) as

$$\begin{aligned}\mathbf{Q}_h &= \left(\mathbf{V}_h - \mathbf{Z}_h\boldsymbol{\Theta}_h - \mathbf{X}_h\boldsymbol{\eta}\right)'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\left(\mathbf{V}_h - \mathbf{Z}_h\boldsymbol{\Theta}_h - \mathbf{X}_h\boldsymbol{\eta}\right) \\ &= \left(\mathbf{V}_h - \mathbf{X}_h\boldsymbol{\eta}\right)'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\left(\mathbf{V}_h - \mathbf{X}_h\boldsymbol{\eta}\right) + \boldsymbol{\Theta}_h'\mathbf{Z}_h'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{Z}_h\boldsymbol{\Theta}_h \\ &\quad - 2\boldsymbol{\Theta}_h'\mathbf{Z}_h'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\left(\mathbf{V}_h - \mathbf{X}_h\boldsymbol{\eta}\right)\end{aligned} \tag{A2}$$

Now,

$$\begin{aligned}\frac{\partial \log \mathrm{L}}{\partial \boldsymbol{\Theta}_h} &= \frac{1}{2}\frac{\partial \mathbf{Q}_h}{\partial \boldsymbol{\Theta}_h} \\ &= -\frac{1}{2}\left[2\mathbf{Z}_h'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{Z}_h\boldsymbol{\Theta}_h - 2\mathbf{Z}_h'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\left(\mathbf{V}_h - \mathbf{X}_h\boldsymbol{\eta}\right)\right]\end{aligned} \tag{A3}$$

Setting this derivative to zero and solving for the maximising value $\hat{\boldsymbol{\Theta}}_h$ gives

$$\begin{aligned}\hat{\boldsymbol{\Theta}}_h &= \left[\mathbf{Z}_h'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{Z}_h\right]^{-1}\mathbf{Z}_h'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\left(\mathbf{V}_h - \mathbf{X}_h\boldsymbol{\eta}\right) \\ &= \left[\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right]^{-1}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\left(\mathbf{V}_h - \mathbf{X}_h\boldsymbol{\eta}\right) \\ &= \left(I_n \otimes \frac{1}{M_h} z_h'\right)\left(\mathbf{V}_h - \mathbf{X}_h\boldsymbol{\eta}\right)\end{aligned} \tag{A4}$$

Considering the i^{th} row in equation (A4)

$$\hat{\theta}_{ih} = \frac{1}{M_h} z_h'(v_{ih} - x_h \eta_i) = \bar{v}_{ih} - \bar{x}_h \eta_i \qquad \text{(A5)}$$

where $\bar{v}_{ih}$ is the average expenditure on commodity i for all households of type h and $\bar{x}_h$ is the average income of all h-type households. The result in (A5) is an important one. It means that $\hat{\theta}_{ih}$'s do not depend on $\mathbf{\Omega}_h$ and can be computed at the end of the maximum likelihood algorithm, after we have estimated $\mathbf{\Omega}_h$ and η_i.

Let $\bar{\mathbf{V}}_h' = (\bar{v}_{1h}, \bar{v}_{2h}, \ldots, \bar{v}_{nh})$ and $\bar{X}_h = I_n \otimes \bar{x}_h$. Then,

$$\hat{\mathbf{\Theta}}_h = \bar{\mathbf{V}}_h - \bar{\mathbf{X}}_h \boldsymbol{\eta} \qquad \text{(A6)}$$

Substituting (A6) into (A2) yields

$$\begin{aligned} \mathbf{Q}_h &= \left[\left(\mathbf{V}_h - \mathbf{Z}_h \bar{\mathbf{V}}_h\right) - \left(\mathbf{X}_h - \mathbf{Z}_h \bar{\mathbf{X}}_h\right)\boldsymbol{\eta}\right]' \left(\mathbf{\Omega}_h^{-1} \otimes I_{M_h}\right)\left[\left(\mathbf{V}_h - \mathbf{Z}_h \bar{\mathbf{V}}_h\right) - \left(\mathbf{X}_h - \mathbf{Z}_h \bar{\mathbf{X}}_h\right)\boldsymbol{\eta}\right] \\ &= \left(\mathbf{V}_h^* - \mathbf{X}_h^* \boldsymbol{\eta}\right)' \left(\mathbf{\Omega}_h^{-1} \otimes I_{M_h}\right)\left(\mathbf{V}_h^* - \mathbf{X}_h^* \boldsymbol{\eta}\right) \end{aligned} \qquad \text{(A7)}$$

where $\mathbf{V}_h^* = \mathbf{V}_h - \mathbf{Z}_h \bar{\mathbf{V}}_h$ is a vector of expenditures expressed in terms of deviations from the mean expenditures for each commodity and household type, and $\mathbf{X}_h^* = \mathbf{X}_h - \mathbf{Z}_h \bar{\mathbf{X}}_h$ is a vector of incomes expressed in terms of deviations from the mean incomes for each household type. The concentrated log-likelihood function can now be written as

$$\begin{aligned} \log \mathrm{L}^* &= -\frac{nM}{2}\log(2\pi) - \frac{1}{2}\sum_{h=1}^{H} M_h \log\left|\mathbf{\Omega}_h\right| - \frac{1}{2}\sum_{h=1}^{H}\left(\mathbf{V}_h^* - \mathbf{X}_h^* \boldsymbol{\eta}\right)' \left(\mathbf{\Omega}_h^{-1} \otimes I_{M_h}\right)\left(\mathbf{V}_h^* - \mathbf{X}_h^* \boldsymbol{\eta}\right) \\ &= -\frac{nM}{2}\log(2\pi) + \frac{1}{2}\sum_{h=1}^{H} M_h \log\left|\mathbf{\Omega}_h^{-1}\right| - \frac{1}{2}\sum_{h=1}^{H} tr\left[\mathbf{W}_h \mathbf{\Omega}_h^{-1}\right] \end{aligned} \qquad \text{(A8)}$$

where $\mathbf{W}_h$ is an $(n \times n)$ matrix of $(i,j)^{th}$ element given by[9]

[9] See Judge, et.al. (1988, p.553) for details of the two alternative specifications in (A8). Judge, et.al. also give details on how to differentiate (A8) with respect to $\mathbf{\Omega}_h^{-1}$.

$$\left[\mathbf{W}_h\right]_{ij} = (v_{ih}^* - x_h^*\eta_i)'(v_{jh}^* - x_h^*\eta_j)\,. \tag{A9}$$

Differentiation with respect to $\boldsymbol{\Omega}_h^{-1}$ yields

$$\frac{\partial \log \mathbf{L}^*}{\partial \boldsymbol{\Omega}_h^{-1}} = \frac{M_h}{2}\boldsymbol{\Omega}_h - \frac{1}{2}\mathbf{W}_h \tag{A10}$$

Setting this derivative equal to zero yields the maximum likelihood estimator for $\boldsymbol{\Omega}_h$ given η_i as follows:

$$\hat{\boldsymbol{\Omega}}_h = \frac{1}{M_h}\mathbf{W}_h \tag{A11}$$

To find an expression for the maximum likelihood estimator for $\boldsymbol{\eta}$, given $\boldsymbol{\Omega}_h$, we return to the last term in (A8) and rewrite it as

$$\begin{aligned}\sum_{h=1}^{H}\mathbf{Q}_h^* &= \sum_{h=1}^{H}\left(\mathbf{V}_h^* - \mathbf{X}_h^*\boldsymbol{\eta}\right)'\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\left(\mathbf{V}_h^* - \mathbf{X}_h^*\boldsymbol{\eta}\right)\\ &= \sum_{h=1}^{H}\left[\mathbf{V}_h^{*\prime}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{V}_h^* + \boldsymbol{\eta}'\mathbf{X}_h^{*\prime}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{X}_h^*\boldsymbol{\eta} - 2\boldsymbol{\eta}'\mathbf{X}_h^{*\prime}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{V}_h^*\right]\end{aligned} \tag{A12}$$

Now,

$$\begin{aligned}\frac{\partial \log \mathrm{L}^*}{\partial \boldsymbol{\eta}} &= -\frac{1}{2}\sum_{h=1}^{H}\frac{\partial \mathbf{Q}_h^*}{\partial \boldsymbol{\eta}}\\ &= -\frac{1}{2}\sum_{h=1}^{H}\left[2\mathbf{X}_h^{*\prime}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{X}_h^*\boldsymbol{\eta} - 2\mathbf{X}_h^{*\prime}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{V}_h^*\right]\end{aligned} \tag{A13}$$

Setting this quantity equal to zero yields

$$\left[\sum_{h=1}^{H}\mathbf{X}_h^{*\prime}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{X}_h^*\right]\hat{\boldsymbol{\eta}} = \sum_{h=1}^{H}\mathbf{X}_h^{*\prime}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{V}_h^* \tag{A14}$$

Now, using obvious notation, it can be shown that

$$\mathbf{X}_h^{*\prime}\left(\boldsymbol{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{X}_h^* = x_h^{*\prime}x_h^*\boldsymbol{\Omega}_h^{-1} \tag{A15}$$

and

$$\mathbf{X}_h^{*\prime}\left(\mathbf{\Omega}_h^{-1}\otimes I_{M_h}\right)\mathbf{V}_h^{*}=\left(\mathbf{\Omega}_h^{-1}\otimes x_h^{*\prime}\right)\mathbf{V}_h^{*}\ = x_h^{*\prime}x_h^{*}\mathbf{\Omega}_h^{-1}\hat{\boldsymbol{\eta}}^h \qquad \text{(A16)}$$

where $\hat{\boldsymbol{\eta}}^h=\left[I_n\otimes\left(x_h^{*\prime}x_h^{*}\right)^{-1}x_h^{*\prime}\right]\mathbf{V}_h^{*}$ is the OLS estimator for $\boldsymbol{\eta}$ from observations corresponding only to the h-type households. The i^{th} element in $\hat{\boldsymbol{\eta}}$ is given by $\hat{\boldsymbol{\eta}}_i^h=\left(x_h^{*\prime}x_h^{*}\right)^{-1}x_h^{*\prime}v_{ih}^{*}$. Substituting (A15) and (A16) into (A14) yields

$$\hat{\boldsymbol{\eta}}=\left[\sum_{h=1}^{H}\left(x_h^{*\prime}x_h^{*}\right)\mathbf{\Omega}_h^{-1}\right]^{-1}\sum_{h=1}^{H}\left(x_h^{*\prime}x_h^{*}\right)\mathbf{\Omega}_h^{-1}\hat{\boldsymbol{\eta}}^h \qquad \text{(A17)}$$

Conditional on $\mathbf{\Omega}_h$, the maximum likelihood estimator for $\boldsymbol{\eta}$ is given by a matrix-weighted average of the h-type household OLS estimators $\hat{\boldsymbol{\eta}}^h$ with weights given by $\left(x_h^{*\prime}x_h^{*}\right)\mathbf{\Omega}_h^{-1}$. Maximum likelihood estimators for all the parameters in $\mathbf{\Theta}_h$, $\mathbf{\Omega}_h$ and $\boldsymbol{\eta}$ are given by the simultaneous solution of equations (A5), (A11) and (A17), which form the basis of the iterative estimation procedure described in Section 3.1.

Derivation of the Asymptotic Covariance Matrices

To derive the covariance matrix expressions given in Section 4, the second derivatives of the log-likelihood function specified in (A1) are required. They are:

$$\frac{\partial^2\log\mathbf{L}}{\partial\mathbf{\Theta}_h\partial\mathbf{\Theta}_h'}=-\mathbf{Z}_h(\mathbf{\Omega}_h^{-1}\otimes I_{M_h})\mathbf{Z}_h\ =-\ M_h\mathbf{\Omega}_h^{-1} \qquad \text{(A18)}$$

$$\frac{\partial^2\log\mathrm{L}}{\partial\mathbf{\Theta}_h\partial\mathbf{\Theta}_k'}=0 \qquad (h\neq k) \qquad \text{(A19)}$$

$$\frac{\partial^2\log\mathbf{L}}{\partial\mathbf{\Theta}_h\partial\boldsymbol{\eta}'}=-\mathbf{Z}_h'(\mathbf{\Omega}_h^{-1}\otimes I_{M_h})\mathbf{X}_h=-\ M_h\bar{x}_h\mathbf{\Omega}_h^{-1} \qquad \text{(A20)}$$

$$\frac{\partial^2\log\mathbf{L}}{\partial\boldsymbol{\eta}\partial\boldsymbol{\eta}'}=-\sum_{h=1}^{H}\mathbf{X}_h'(\mathbf{\Omega}_h^{-1}\otimes I_{M_h})\mathbf{X}_h\ =-\sum_{h=1}^{H}x_h'x_h\mathbf{\Omega}_h^{-1} \qquad \text{(A21)}$$

It can be shown that the expectations of the cross partial derivatives with respect to (θ_{ih}, η_i and the elements of $\mathbf{\Omega}_h$ are zero. Thus the information matrix is block diagonal, and, providing interest is not on the standard errors of the maximum likelihood estimator of $\mathbf{\Omega}_h$, concern may be confined to the derivatives (A18) – (A21).

Specifically, let $\mathbf{\Theta}' = \left(\mathbf{\Theta}_1', \mathbf{\Theta}_2', \ldots, \mathbf{\Theta}_H', \boldsymbol{\eta}'\right)$, then

$$-\frac{\partial^2 \log \mathrm{L}}{\partial \mathbf{\Theta} \partial \mathbf{\Theta}'} = \begin{bmatrix} M_1 \mathbf{\Omega}_1^{-1} & 0 & \cdots & 0 & M_1 \bar{x}_1 \mathbf{\Omega}_1^{-1} \\ 0 & M_2 \mathbf{\Omega}_2^{-1} & \cdots & 0 & M_2 \bar{x}_2 \mathbf{\Omega}_2^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_H \mathbf{\Omega}_H^{-1} & M_H \bar{x}_H \mathbf{\Omega}_H^{-1} \\ M_1 \bar{x}_1 \mathbf{\Omega}_1^{-1} & M_2 \bar{x}_2 \mathbf{\Omega}_2^{-1} & \cdots & M_H \bar{x}_H \mathbf{\Omega}_H^{-1} & \sum_{h=1}^{H} x_h' x_h \mathbf{\Omega}_h^{-1} \end{bmatrix} \qquad \text{(A22)}$$

Since this matrix does not contain any stochastic elements, the information matrix obtained by taking expectations of (A22) is the same as (A22). Let $D = \sum_{h=1}^{H} \left[\left(x_h' x_h - M_h \bar{x}_h^2 \right) \mathbf{\Omega}_h^{-1} \right]$, $\bar{x} = \left(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_H \right)'$ and

$$A = \begin{bmatrix} M_1^{-1} \mathbf{\Omega}_1 & & & \\ & M_2^{-1} \mathbf{\Omega}_2 & & \\ & & \ddots & \\ & & & M_H^{-1} \mathbf{\Omega}_H \end{bmatrix} \qquad \text{(A23)}$$

Using results on the partitioned inverse of a matrix and using $V(.)$ to denote the asymptotic covariance matrix, it can be shown that

$$\begin{aligned} V(\hat{\mathbf{\Theta}}) &= \left[E\left(-\frac{\partial^2 \log \mathrm{L}}{\partial \mathbf{\Theta} \partial \mathbf{\Theta}'} \right) \right]^{-1} \\ &= \begin{bmatrix} A^{-1} & A^{-1}(\bar{x} \otimes I) \\ (\bar{x}' \otimes I) A^{-1} & \sum x_h x_h' \mathbf{\Omega}_h^{-1} \end{bmatrix}^{-1} \qquad \text{(A24)} \\ &= \begin{bmatrix} A + \bar{x}\bar{x}' \otimes D^{-1} & -\bar{x} \otimes D^{-1} \\ -\bar{x} \otimes D^{-1} & D^{-1} \end{bmatrix} \end{aligned}$$

The relevant variance components from (A24) are $V(\hat{\boldsymbol{\Theta}}_h) = M_h^{-1}\boldsymbol{\Omega}_h + \bar{x}_h^2 D^{-1}$ and $V(\hat{\boldsymbol{\eta}}) = D^{-1}$.

We now consider the asymptotic variances and covariances for the estimators of the original parameters b, a_h, b_i and a_{ih} defined in equations (9) – (12). From (9), $V(\hat{b}) = z'D^{-1}z$ where $z = (1,1,\ldots,1)'$. Let $\mathbf{B} = (b_1, b_2, \ldots, b_n)'$. Then

$$V(\hat{\mathbf{B}}) = \left(\frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}'}\right) D^{-1} \left(\frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}'}\right)' \qquad \text{(A25)}$$

Now,

$$\frac{\partial b_i}{\partial \eta_j} = \begin{cases} \dfrac{1}{b}\left(1 - \dfrac{\eta_i}{b}\right) & \textit{for} \quad i = j \\ \dfrac{1}{b}\left(-\dfrac{\eta_i}{b}\right) & \textit{for} \quad i \neq j \end{cases} \qquad \text{(A26)}$$

Let $C = I_n - \frac{1}{b}\boldsymbol{\eta} z'$. From (A25) and (A26), it follows that $\frac{\partial \mathbf{B}}{\partial \boldsymbol{\eta}} = \frac{1}{b}C$ and $V(\hat{\mathbf{B}}) = \frac{1}{b^2} C D^{-1} C'$.

Consider now the covariance matrix for the $\hat{a}_{ih}$. Let $\boldsymbol{\alpha}_h = \left(a_{1h}, a_{2h}, \ldots, a_{nh}\right)'$. By definition, we have

$$V(\hat{\boldsymbol{\alpha}}_h) = \begin{bmatrix} \dfrac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\Theta}_h'} & \dfrac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\eta}'} \end{bmatrix} V\begin{pmatrix} \hat{\boldsymbol{\Theta}}_h \\ \hat{\boldsymbol{\eta}} \end{pmatrix} \begin{bmatrix} \left(\dfrac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\Theta}_h'}\right)' \\ \left(\dfrac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\eta}'}\right)' \end{bmatrix} \qquad \text{(A27)}$$

Now, $\frac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\Theta}_h'} = C^*$ and $\frac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\eta}'} = a_h C^*$ where $C^* = I_n + \frac{1}{1-b}\boldsymbol{\eta} z'$. Thus,

$$\begin{aligned} V(\hat{\boldsymbol{\alpha}}_h) &= \begin{bmatrix} C^* & a_h C^* \end{bmatrix} \begin{bmatrix} M_h^{-1}\boldsymbol{\Omega}_h + \bar{x}_h^2 D^{-1} & -\bar{x}_h D^{-1} \\ -\bar{x}_h D^{-1} & D^{-1} \end{bmatrix} \begin{bmatrix} C^{*\prime} \\ a_h C^{*\prime} \end{bmatrix} \\ &= C^* \left[M_h^{-1}\boldsymbol{\Omega}_h + \left(\bar{x}_h - a_h\right)^2 D^{-1} \right] C^{*\prime} \end{aligned} \qquad \text{(A28)}$$

Noting that $\hat{a}_h = z'\hat{a}_{ih}$, we also have

$$V(\hat{a}_h) = z'C^*\left[M_h^{-1}\mathbf{\Omega}_h + \left(\overline{x}_h - a_h\right)^2 D^{-1}\right]C^{*\prime}z \qquad \text{(A29)}$$

Turning now to an expression for the variance estimator of the commodity specific scales $\hat{s}_{ih} = \frac{\hat{a}_{ih}}{\hat{a}_{ir}}$, we obtain

$$\begin{aligned} V(\hat{s}_{ih}) &= \left(\frac{\partial s_{ih}}{\partial a_{ih}}\right)^2 V\left(\hat{a}_{ih}\right) + \left(\frac{\partial s_{ih}}{\partial a_{ir}}\right)^2 V\left(\hat{a}_{ir}\right) + 2\left(\frac{\partial s_{ih}}{\partial a_{ih}}\right)\left(\frac{\partial s_{ih}}{\partial a_{ir}}\right)\mathrm{cov}\left(\hat{a}_{ih}, \hat{a}_{ir}\right) \\ &= \frac{1}{a_{ir}^2} V\left(\hat{a}_{ih}\right) + \frac{a_{ih}^2}{a_{ir}^4} V\left(\hat{a}_{ir}\right) - \frac{2a_{ih}}{a_{ir}^3}\mathrm{cov}\left(\hat{a}_{ih}, \hat{a}_{ir}\right) \end{aligned} \qquad \text{(A30)}$$

The elements $V(\hat{a}_{ih})$ and $V(\hat{a}_{ir})$ are given by the appropriately selected diagonal elements in (A30) and its counterpart for the reference household. The elements $\mathrm{cov}(\hat{a}_{ih}, \hat{a}_{ir})$ are given by the diagonal elements of

$$\mathrm{cov}\left(\hat{\boldsymbol{\alpha}}_h, \hat{\boldsymbol{\alpha}}_r\right) = \begin{bmatrix} \frac{\partial \boldsymbol{\alpha}_r}{\partial \boldsymbol{\Theta}_h'} & \frac{\partial \boldsymbol{\alpha}_r}{\partial \boldsymbol{\Theta}_r'} & \frac{\partial \boldsymbol{\alpha}_r}{\partial \boldsymbol{\eta}'} \end{bmatrix} V\begin{pmatrix} \hat{\boldsymbol{\Theta}}_h \\ \hat{\boldsymbol{\Theta}}_r \\ \hat{\boldsymbol{\eta}} \end{pmatrix} \begin{bmatrix} \left(\frac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\Theta}_h'}\right)' \\ \left(\frac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\Theta}_r'}\right)' \\ \left(\frac{\partial \boldsymbol{\alpha}_h}{\partial \boldsymbol{\eta}'}\right)' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & C^* & a_r C^* \end{bmatrix} \begin{bmatrix} M_h^{-1}\mathbf{\Omega}_h + \overline{x}_h^2 D^{-1} & \overline{x}_h \overline{x}_r D^{-1} & -\overline{x}_h D^{-1} \\ \overline{x}_h \overline{x}_r D^{-1} & M_h^{-1}\mathbf{\Omega}_r + \overline{x}_r^2 D^{-1} & -\overline{x}_r D^{-1} \\ -\overline{x}_h D^{-1} & -\overline{x}_r D^{-1} & D^{-1} \end{bmatrix} \begin{bmatrix} C^{*\prime} \\ 0 \\ a_h C^{*\prime} \end{bmatrix} \qquad \text{(A31)}$$

$$= \left(\overline{x}_h - a_h\right)\left(\overline{x}_r - a_r\right) C^* D^{-1} C^{*\prime}$$

These results are summarised in Section 3.2.